“Aggregate Planning with PuLP”

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Implicit Formulation for the Base Problem

$Sets$: i $= \{1, 2 \ldots \quad N\}$
$t \quad = \{0, 1 \ldots \mathrm{T}\}$

$Parameters$:

d_{it} : Demand for product i at period t
h : Hiring cost
f : Firing cost
c_i : Inventory cost of item i
w_{init} : Initial workforce
ii_i : Initial inventory of item i
pc_i : Production cost of item i
p : Default productivity of a worker
pwf_i : Unit workforce requirement of item i
r : Regular time employee cost
o : Regular time employee cost
o_{lim} : Demand for the product i at period t

$Decision\ variables$:

x_{it} : $Amount\ of\ product\ i\ produced\ at\ period\ t$
I_{it} : $Stock\ level\ of\ product\ i\ at\ the\ end\ of\ period\ t$
H_t : $Amount\ of\ hired\ workers\ at\ period\ t$
F_t : $Amount\ of\ fired\ workers\ at\ period\ t$
W_t : $Workforce\ level\ at\ period\ t$
O_t : $Amount\ of\ overtime\ workers\ at\ period\ t$

$$min \sum_{t=1}^{T} \sum_{i=1}^{N} pc_i . x_{it} + \sum_{t=1}^{T} \sum_{i=1}^{N} c_i . x_{it} + \sum_{t=1}^{T} r.W_t + o.O_t + \sum_{t=1}^{T} h.H_t + f.F_t$$

(Production costs + Inventory costs + Wages + Hiring/firing costs)

$$st \qquad \begin{aligned} & I_{i,t-1} + x_{it} = d_{it} + I_{it} && ,\forall_{it} && (Inventory\ balance) \\ & W_t = W_{t-1} + H_t + F_t && ,\forall_t && (Workforce\ link) \\ & O_t \le W_t && ,\forall_t && (Overtime\ limitation) \\ & p.W_t + o_{lim}.O_t \ \ge \sum_{i=1}^{N} pwf_i . x_{it} && ,\forall_t && (Workforce\ limitation) \\ & I_{i0} = \ ii_i && ,\forall_i && (Initial\ inventory\ levels) \\ & W_0 \ = w_{init} && && (Initial\ workforce) \end{aligned}$$

$$\begin{aligned} & x_{it} \ge 0, I_{it} \ \ge 0 && ,\forall_{it} \\ & H_t \ \ge 0, F_t \ \ge 0, W_t \ge 0, O_t \ \ge 0 && ,\forall_t \end{aligned}$$

Solutions

Optimal Solution for the Base Problem

Optimal value of the objective function: $*3,143,976.38*

F_1 = 11.3889	$I_{1,1}$ = 0.0	O_1 = 0.0	$X_{1,1}$ = 20.0
F_2 = 2.94444	$I_{1,2}$ = 0.0	O_2 = 0.0	$X_{1,2}$ = 20.0
F_3 = 0.0	$I_{1,3}$ = 0.0	O_3 = 0.0	$X_{1,3}$ = 26.0
F_4 = 0.0	$I_{1,4}$ = 0.0	O_4 = 0.0	$X_{1,4}$ = 24.0
F_5 = 0.0	$I_{1,5}$ = 0.0	O_5 = 0.0	$X_{1,5}$ = 18.0
F_6 = 0.0	$I_{1,6}$ = 12.7895	O_6 = 0.0	$X_{1,6}$ = 22.7895
F_7 = 0.0	$I_{1,7}$ = 10.7895	O_7 = 0.0	$X_{1,7}$ = 20.0
F_8 = 0.0	$I_{1,8}$ = 8.0	O_8 = 0.0	$X_{1,8}$ = 17.2105
F_9 = 0.0	$I_{1,9}$ = 17.5013	O_9 = 0.0	$X_{1,9}$ = 27.5013
F_{10} = 0.0	$I_{1,10}$ = 10.2132	O_{10} = 0.0	$X_{1,10}$ = 24.7118
F_{11} = 0.0	$I_{1,11}$ = 0.527632	O_{11} = 75.725	$X_{1,11}$ = 24.3145
F_{12} = 0.0	$I_{1,12}$ = 0.0	O_{12} = 75.725	$X_{1,12}$ = 35.4724
H_1 = 0.0	$I_{1,0}$ = 3.0	W_0 = 86.0	$X_{1,1}$ = 11.0
H_2 = 0.0	$I_{1,1}$ = 0.0	W_1 = 74.6111	$X_{1,2}$ = 10.0
H_3 = 0.0	$I_{1,2}$ = 0.0	W_2 = 71.6667	$X_{1,3}$ = 4.0
H_4 = 0.0	$I_{1,3}$ = 0.0	W_3 = 71.6667	$X_{1,4}$ = 4.0
H_5 = 0.0	$I_{1,4}$ = 0.0	W_4 = 71.6667	$X_{1,5}$ = 6.0
H_6 = 0.0	$I_{1,5}$ = 0.0	W_5 = 71.6667	$X_{1,6}$ = 8.0
H_7 = 0.0	$I_{1,6}$ = 0.0	W_6 = 71.6667	$X_{1,7}$ = 10.0
H_8 = 0.0	$I_{1,7}$ = 0.0	W_7 = 71.6667	$X_{1,8}$ = 12.0
H_9 = 4.05833	$I_{1,8}$ = 0.0	W_8 = 71.6667	$X_{1,9}$ = 6.0
H_{10} = 0.0	$I_{1,9}$ = 0.0	W_9 = 75.725	$X_{1,10}$ = 8.0
H_{11} = 0.0	$I_{1,10}$ = 0.0	W_{10} = 75.725	$X_{1,11}$ = 14.0
H_{12} = 0.0	$I_{1,11}$ = 0.0	W_{11} = 75.725	$X_{1,12}$ = 6.0
$I_{1,0}$ = 8.0	$I_{1,12}$ = 0.0	W_{12} = 75.725	

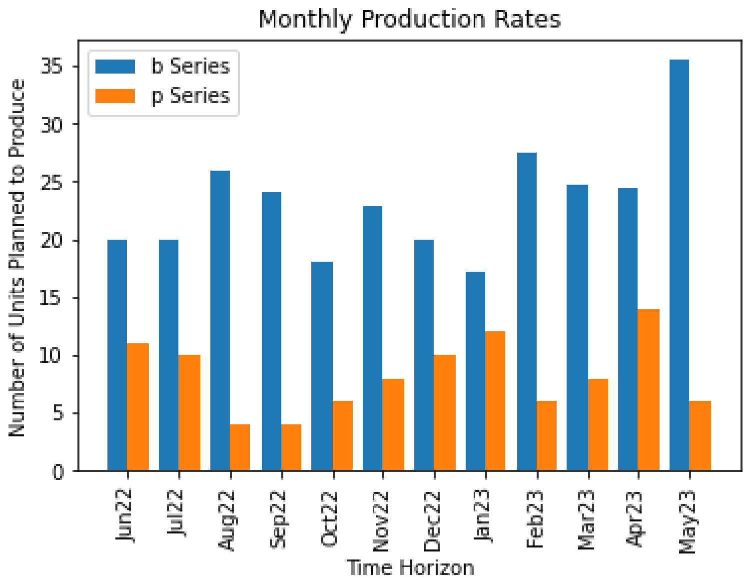

Figure 1: Optimal Monthly Production Plan for the Base Model

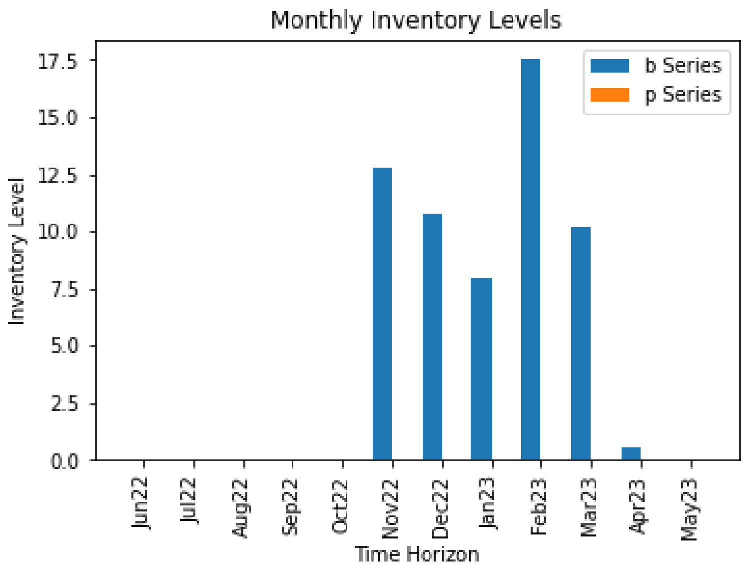

Figure 2: Optimal Inventory Levels for the Base Model

According to the optimal values of the decision variables from the above model, “modified constant workforce plan” seems like the most suitable strategy for the company since the amount of hired and fired workers are zero at almost every period. The lay-offs in the first two months imply that there is a labor surplus initially. The optimal solution suggests that, after these lay-offs, only one hiring activity in February is enough for yearly workforce adjustment. No lay-offs after the first two months is also reasonable from the qualitative aspect when the reputation of the company as an employer is considered. Another insight from the results is that assigning overtime schedules to workers is not a profitable strategy.

From *Figure 2*, the obviously distinct strategy for the two product types can be observed. For pro series, a more volatile production plan would be more beneficial, probably due to its high inventory cost. In other words, there is no sense in producing a pro series caravan ahead of time. When it comes to the basic series, the company faces the requirement of high inventory levels between November and April. This outcome can be associated with the high demand forecasts for the last quarter (March, April, and May) and can be regarded as a preparation for this dense period.

Extension 1: Inclusion of the Temporary Workers

Introducing the extra workforce options for December and January to the model did not cause any changes in the optimal solution apart from the addition of two new decision variables:

$$E_7 = 0$$
$$E_8 = 0$$

These variables represent the amount of extra workforce brought to the facility at the 7^{th} and 8^{th} periods. It can be seen that there is no need for extra workforce in these periods.

Extension 2: Inclusion of the Space Constraint

In this part, a new type of decision variable, representing the amount of extra space rented in each period *(E_t)*, is introduced to the model along with several related parameters and a space constraint. Taking the decision variable values of the new optimal solution into consideration, the extra capacity rental can be regarded as an inexpensive option, since there is no significant change in the remaining decision variables. The new optimum is obtained by just adding the required rental space penalty to the total cost. After the addition, the new optimal solution is *$3,155,116.38*.

Updated Solution for Extension 2

$E_1 = 960.0$
$E_2 = 900.0$
$E_3 = 780.0$
$E_4 = 700.0$
$E_5 = 580.0$
$E_6 = 891.579$
$E_7 = 900.0$
$E_8 = 908.421$
$E_9 = 960.053$
$E_{10} = 968.474$
$E_{11} = 1312.58$
$E_{12} = 1278.89$
$F_1 = 11.3889$
$F_2 = 2.94444$
$F_3 = 0.0$
$F_4 = 0.0$
$F_5 = 0.0$
$F_6 = 0.0$
$F_7 = 0.0$
$F_8 = 0.0$
$F_9 = 0.0$
$F_{10} = 0.0$
$F_{11} = 0.0$
$F_{12} = 0.0$
$H_1 = 0.0$
$H_2 = 0.0$
$H_3 = 0.0$
$H_4 = 0.0$
$H_5 = 0.0$
$H_6 = 0.0$
$H_7 = 0.0$
$H_8 = 0.0$
$H_9 = 4.05833$
$H_{10} = 0.0$
$H_{11} = 0.0$
$H_{12} = 0.0$
$I_{1,0} = 8.0$
$I_{1,1} = 0.0$
$I_{1,2} = 0.0$
$I_{1,3} = 0.0$
$I_{1,4} = 0.0$
$I_{1,5} = 0.0$
$I_{1,6} = 12.7895$
$I_{1,7} = 10.7895$
$I_{1,8} = 8.0$
$I_{1,9} = 17.5013$
$I_{1,10} = 10.2132$
$I_{1,11} = 0.527632$
$I_{1,12} = 0.0$
$I_{2,0} = 3.0$
$I_{2,1} = 0.0$
$I_{2,2} = 0.0$
$I_{2,3} = 0.0$
$I_{2,4} = 0.0$
$I_{2,5} = 0.0$
$I_{2,6} = 0.0$
$I_{2,7} = 0.0$
$I_{2,8} = 0.0$
$I_{2,9} = 0.0$
$I_{2,10} = 0.0$
$I_{2,11} = 0.0$
$I_{2,12} = 0.0$
$O_1 = 0.0$
$O_2 = 0.0$
$O_3 = 0.0$
$O_4 = 0.0$
$O_5 = 0.0$
$O_6 = 0.0$
$O_7 = 0.0$
$O_8 = 0.0$
$O_9 = 0.0$
$O_{10} = 0.0$
$O_{11} = 75.725$
$O_{12} = 75.725$
$W_0 = 86.0$
$W_1 = 74.6111$
$W_2 = 71.6667$
$W_3 = 71.6667$
$W_4 = 71.6667$
$W_5 = 71.6667$
$W_6 = 71.6667$
$W_7 = 71.6667$
$W_8 = 71.6667$
$W_9 = 75.725$
$W_{10} = 75.725$
$W_{11} = 75.725$
$W_{12} = 75.725$
$X_{1,1} = 20.0$
$X_{1,2} = 20.0$
$X_{1,3} = 26.0$
$X_{1,4} = 24.0$
$X_{1,5} = 18.0$
$X_{1,6} = 22.7895$
$X_{1,7} = 20.0$
$X_{1,8} = 17.2105$
$X_{1,9} = 27.5013$
$X_{1,10} = 24.7118$
$X_{1,11} = 24.3145$
$X_{1,12} = 35.4724$
$X_{2,1} = 11.0$
$X_{2,2} = 10.0$
$X_{2,3} = 4.0$
$X_{2,4} = 4.0$
$X_{2,5} = 6.0$
$X_{2,6} = 8.0$
$X_{2,7} = 10.0$
$X_{2,8} = 12.0$
$X_{2,9} = 6.0$
$X_{2,10} = 8.0$
$X_{2,11} = 14.0$
$X_{2,12} = 6.0$

Sensitivity Analysis

#	Constraint	Shadow Price	Slack
1	$-40\,O_1 - 180\,W_1 + 380\,X_{1,01} + 530\,X_{2,01} \leq 0$	-2,33	1,82E-12
2	$-40\,O_2 - 180\,W_2 + 380\,X_{1,02} + 530\,X_{2,02} \leq 0$	-0,87	-9,1E-13
3	$-40\,O_3 - 180\,W_3 + 380\,X_{1,03} + 530\,X_{2,03} \leq 0$	0,00	900
4	$-40\,O_4 - 180\,W_4 + 380\,X_{1,04} + 530\,X_{2,04} \leq 0$	0,00	1660
5	$-40\,O_5 - 180\,W_5 + 380\,X_{1,05} + 530\,X_{2,05} \leq 0$	0,00	2880
6	$-40\,O_6 - 180\,W_6 + 380\,X_{1,06} + 530\,X_{2,06} \leq 0$	-0,42	-1,8E-12
7	$-40\,O_7 - 180\,W_7 + 380\,X_{1,07} + 530\,X_{2,07} \leq 0$	-1,08	-9,1E-13
8	$-40\,O_8 - 180\,W_8 + 380\,X_{1,08} + 530\,X_{2,08} \leq 0$	-1,74	0
9	$-40\,O_9 - 180\,W_9 + 380\,X_{1,09} + 530\,X_{2,09} \leq 0$	-2,40	0
10	$-40\,O_{10} - 180\,W_{10} + 380\,X_{1,10} + 530\,X_{2,10} \leq 0$	-3,06	1,82E-12
11	$-40\,O_{11} - 180\,W_{11} + 380\,X_{1,11} + 530\,X_{2,11} \leq 0$	-3,71	1,82E-12
12	$-40\,O_{12} - 180\,W_{12} + 380\,X_{1,12} + 530\,X_{2,12} \leq 0$	-4,37	1,82E-12

Table 1: Shadow Prices and Slack Variables of the Capacity Constraints

From the table above, it can be concluded that, excluding the 3rd, 4th, and 5th months, increases in unit workforce requirement of caravans are potential threats for the feasibility of the current optimal solution, since their slack values are 0 or close to 0, which implies that in these months the resources are used almost at full capacity. For these binding constraints, increasing resources improves the optimal value by their corresponding shadow price value.

The optimal solution suggested above can be regarded as sensitive, and consequently risky. It can be improved by the introduction of suitable "chance constraints" with a 90% probability to obtain a more "robust" optimization model.